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Spin^T structure and Dirac operator on Riemannian manifolds

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Abstract

In this paper, we describe the group $\mathrm{Spin}^T(n)$ and give some properties of this group. We construct Spin^T spinor bundle $\mathbb{S}$ by means of the spinor representation of the group $\mathrm{Spin}^T(n)$ and define covariant derivative operator and Dirac operator on $\mathbb{S}$. Finally, Schrödinger-Lichnerowicz-type formula is derived by using these operators.

Key Words Spinor bundle, the group $\mathrm{Spin}^T(n)$, Dirac operator, Schrödinger-Lichnerowicz-type formula.
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1 Introduction

Spin and Spin^c structures is effective tool to study the geometry and topology of manifolds, especially in dimension four. Spin and Spin^c manifolds have been studied extensively in [2, 3, 4, 5]. For any compact Lie group G the Spin^G structure have been studied in [1]. However, the spinor representation is replaced by a hyperkahler manifold, also called target manifold. In this paper, we define the Lie group $\mathrm{Spin}^T(n)$ as a quotient group by taking $G = S^1 \times S^1$. The groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}^c(n)$ are the subset of $\mathrm{Spin}^T(n)$. We define Spin^T structure on any Riemannian manifold. The spinor representation of $\mathrm{Spin}^T(n)$ is defined by the help of the spinor representation of $\mathrm{Spin}(n)$. By using the spinor representation of $\mathrm{Spin}^T(n)$ we construct the Spin^T spinor bundle $\mathbb{S}$. Finally, we give Schrödinger-Lichnerowicz-type formula by using covariant derivative operator and Dirac operator on $\mathbb{S}$.

This paper is organized as follows. We begin with a section introducing the group $\mathrm{Spin}^T(n)$. In the following section, we define Spin^T structure on any Riemannian manifold. The final section is dedicated to the construction of

the spinor bundle $\mathbb{S}$, the study of the Dirac operator associated to Levi-Civita connection ∇ and Schrödinger-Lichnerowicz-type formula.

2 The group $\mathbf{Spin}^T(n)$

Definition 1 *The $Spin^T$ group is defined as*

$$Spin^T(n) := (Spin(n) \times S^1 \times S^1)/\{\pm 1\}.$$

The elements of $\mathrm{Spin}^T(n)$ are thus classes $[g, z_1, z_2]$ of pairs $(g, z_1, z_2) \in Spin(n) \times S^1 \times S^1$ under the equivalence relation

$$(g, z_1, z_2) \sim (-g, -z_1, -z_2).$$

We can define the following homomorphisms:

a. The map $\lambda^T : Spin^T(n) \longrightarrow SO(n)$ is given by $\lambda^T([g, z_1, z_2]) = \lambda(g)$ where the map $\lambda : Spin(n) \to SO(n)$ is the two-fold covering given by $\lambda(g)(v) = gvg^{-1}$.

b. $i : Spin(n) \longrightarrow Spin^T(n)$ is the natural inclusion map $i(g) = [g, 1, 1]$.

c. $j : S^1 \times S^1 \longrightarrow Spin^T(n)$ is the inclusion map $j(z_1, z_2) = [1, z_1, z_2]$.

d. $l : Spin^T(n) \longrightarrow S^1 \times S^1$ is given by $l([g, z_1, z_2]) = (z_1^2, z_1 z_2)$.

e. $p : Spin^T(n) \longrightarrow SO(n) \times S^1 \times S^1$ is given by $p([g, z_1, z_2]) = (\lambda(g), z_1^2, z_1 z_2)$. Hence, $p = \lambda^T \times l$. Here p is a 2-fold covering.

Thus, we obtain the following commutative diagram where the row and the column are exact.

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & S^1 \times S^1 \arrow[d, "j"] \arrow[dr] & & \\
1 \arrow[r] & Spin(n) \arrow[r, "i"] \arrow[dr, "\lambda"] & Spin^T(n) \arrow[r, "l"] \arrow[d, "\lambda^T"] & S^1 \times S^1 \arrow[r] & 1 \\
 & & SO(n) \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}$$

Moreover, we have the following exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^T(n) \xrightarrow{p} SO(n) \times S^1 \times S^1 \longrightarrow 1.$$

Theorem 2 *The group Spin$^T(n)$ is isomorphic to Spin$^c(n) \times S^1$.*

Proof We define the map φ in the following way:

$$\begin{array}{ccc} Spin(n) \times S^1 \times S^1 & \overset{\varphi}{\to} & Spin^c(n) \times S^1 \\ (g, z_1, z_2) & \mapsto & ([g, z_1], z_1 z_2) \end{array}$$

It can be easily shown that φ is a surjective homomorphism and the kernel of φ is $\{(1,1,1), (-1,-1,-1)\}$. Thus, the group $\mathrm{Spin}^T(n)$ is isomorphic to $\mathrm{Spin}^c(n) \times S^1$. $\square$

Since $\mathrm{Spin}(n)$ is contained in the complex Clifford algebra $\mathbb{C}l_n$, the spin representation κ of the group $\mathrm{Spin}(n)$ extends to a $\mathrm{Spin}^T(n)$-representation. For an element $[g, z_1, z_2]$ from $\mathrm{Spin}^T(n)$ and any spinor $\psi \in \Delta_n$, the spinor representation κ^T of $\mathrm{Spin}^T(n)$ is given by

$$\kappa^T[g, z_1, z_2]\psi = z_1^2 z_2 \kappa(g)(\psi).$$

Proposition 3 *If $n = 2k+1$ is odd, then κ^T is irreducible.*

Proof Assume that $\{0\} \neq W \neq \Delta_{2k+1}$ is a Spin^T invariant subspace. Thus, we have $\kappa^T[g, z_1, z_2](W) \subseteq W$. That is, $z_1^2 z_2 \kappa(g)(W) \subseteq W$. In this case, for every $w \in W$ there exists a $w' \in W$ such that $z_1^2 z_2 \kappa(g)(w) = w'$. As $\kappa(g)(w) = \frac{1}{z_1^2 z_2} w' \in W$ and the representation κ of $\mathrm{Spin}(n)$ is irreducible if n is odd, this is a contradiction. The representation κ^T of $\mathrm{Spin}^T(n)$ has to be irreducible for $n = 2k+1$.
$\square$

Proposition 4 *If $n = 2k$ is even, then the spinor space Δ_{2k} decomposes into two subspaces $\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-$.*

Proof We know that the $\mathrm{Spin}(n)$ representation Δ_{2k} decomposes into two subspaces Δ_{2k}^+ and Δ_{2k}^-. Thus, we obtain $z_1^2 z_2 \kappa(g)(\Delta_{2k}^+) \subseteq \Delta_{2k}^+$ and $z_1^2 z_2 \kappa(g)(\Delta_{2k}^-) \subseteq \Delta_{2k}^-$. Namely, $\kappa^T[g, z_1, z_2](\Delta_{2k}^+) \subseteq \Delta_{2k}^+$ and $\kappa^T[g, z_1, z_2](\Delta_{2k}^-) \subseteq \Delta_{2k}^-$. Hence, the $\mathrm{Spin}^T(2k)$ representation Δ_{2k} decomposes into two subspaces Δ_{2k}^+ and Δ_{2k}^-. It can be easily seen that the $\mathrm{Spin}^T(2k)$ representation $\Delta_{2k}^\pm$ is irreducible. $\square$

The Lie algebra of the group $\mathrm{Spin}^T(n)$ is described by

$$\mathfrak{spin}^T(n) = \mathfrak{m}_2 \oplus i\mathbb{R} \oplus i\mathbb{R}.$$

The differential $p_* : \mathfrak{spin}^T(n) \to \mathfrak{so}(n) \oplus i\mathbb{R} \oplus i\mathbb{R}$ is defined by

$$p_*(e_\alpha e_\beta, \lambda i, \mu i) = (2E_{\alpha\beta}, 2\lambda i, (\lambda + \mu) i)$$

where λ and μ are any real numbers and $E_{\alpha\beta}$ is the $n \times n$ matrix with entries $(E_{\alpha\beta})_{\alpha\beta} = -1$, $(E_{\alpha\beta})_{\beta\alpha} = 1$ and all others are equal to zero. The inverse of the differential p_* is given by

$$p_*^{-1}(E_{\alpha\beta}, \lambda i, \mu i) = (\frac{1}{2} e_\alpha e_\beta, \frac{1}{2}\lambda i, (\mu - \frac{1}{2}\lambda) i).$$

3 Spin^T structure

Definition 5 *A Spin^T structure on an oriented Riemannian manifold* (M^n, g) *is a* $Spin^T(n)$ *principal bundle* $P_{Spin^T(n)}$ *together with a smooth map* $\Lambda : P_{Spin^T(n)} \to P_{SO(n)}$ *such that the following diagram commutes:*

$$\begin{tikzcd} P_{Spin^T(n)} \times Spin^T(n) \arrow[r] \arrow[d, "\Lambda\times\lambda^T"] & P_{Spin^T(n)} \arrow[d, "\Lambda"] \\ P_{SO(n)} \times SO(n) \arrow[r] & P_{SO(n)} \end{tikzcd}$$

From above definition we can construct a two-fold covering map

$$\Pi : P_{Spin^T(n)} \to P_{SO(n)} \times P_{S^1\times S^1}.$$

Given a Spin^T structure $(P_{Spin^T(n)}, \Lambda)$, the map $\lambda^T : Spin^T(n) \longrightarrow SO(n)$ induces an isomorphism

$$P_{Spin^T(n)}/S^1 \times S^1 \cong P_{SO(n)}.$$

In similar way, $Spin^T(n)/_{Spin(n)} \cong S^1 \times S^1$ implies the isomorphism

$$P_{Spin^T(n)}/Spin(n) \cong P_{S^1\times S^1}.$$

Note that on account of the inclusion map $i : Spin(n) \to Spin^T(n)$, every spin structure on M induces a Spin^T structure. Similarly, since there exists a inclusion map $Spin^c(n) \to Spin^T(n)$, every Spin^c structure on M induces a Spin^T structure.

4 Spinor bundle and Dirac operator

Let (M^n, g) be an oriented connected Riemannian manifold and $P_{SO(n)} \to M$ the $SO(n)-$principal bundle of positively oriented orthonormal frames. The Levi-Civita connection ∇ on $P_{SO(n)}$ determine a connection $1-$form ω on the principal bundle $P_{SO(n)}$ with values in $\mathfrak{so}(n)$, locally given by

$$\omega^e = \sum_{i<j} g(\nabla e_i, e_j) E_{ij}$$

where $e = \{e_1, \ldots, e_n\}$ is a local section of $P_{SO(n)}$ and E_{ij} is the $n \times n$ matrix with entries $(E_{ij})_{ij} = -1$, $(E_{ij})_{ji} = 1$ and all others are equal to zero.

We fix a connection

$$(A, B) : TP_{S^1\times S^1} \to i\mathbb{R} \oplus i\mathbb{R}$$

on the principal bundle $P_{S^1\times S^1}$. The connections ω and (A, B) induce a connection

$$\omega \times (A, B) : T(P_{SO(n)} \times P_{S^1\times S^1}) \to \mathfrak{so}(n) \oplus i\mathbb{R} \oplus i\mathbb{R}$$

on the fibre product bundle $P_{SO(n)} \times P_{S^1 \times S^1}$. Now we can define a connection $1-$form $\widetilde{\omega \times (A,B)}$ on the principal bundle $P_{Spin^T(n)}$ such that the following diagram commutes:

$$\begin{tikzcd} TP_{Spin^T(n)} \arrow[r, "\widetilde{\omega \times (A,B)}"] \arrow[d, "\Pi_*"] & \mathfrak{spin}^T(n) = \mathfrak{m}_2 \oplus i\mathbb{R} \oplus i\mathbb{R} \arrow[d, "p_*"] \\ T(P_{SO(n)} \times P_{S^1 \times S^1}) \arrow[r, "\omega \times (A,B)"] & \mathfrak{so}(n) \oplus i\mathbb{R} \oplus i\mathbb{R} \end{tikzcd}$$

That is, the equality

$$p_* \circ \widetilde{\omega \times (A,B)} = \omega \times (A,B) \circ \Pi_*$$

holds.

Definition 6 *The spinor bundle of a Spin^T manifold is defined as the associated vector bundle*

$$\mathbb{S} = P_{Spin^T(n)} \times_{\kappa^T} \Delta_n$$

where $\kappa^T : Spin^T(n) \to GL(\Delta_n)$ is the spinor representation of $Spin^T(n)$. In case of $n = 2k$ the spinor bundle splits into the sum of two subbundles $\mathbb{S}^+$ and $\mathbb{S}^-$ such that

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-, \qquad \mathbb{S}^\pm = P_{Spin^T(n)} \times_{\kappa^{T\pm}} \Delta_n^\pm.$$

Any spinor field ψ can be identified with the map $\psi : P_{Spin^T(n)} \to \Delta_n$ satisfying the transformation rule $\psi(pg) = \kappa^T(g^{-1})\psi(p)$. The absolute differential of a section ψ with respect to $\widetilde{\omega \times (A,B)}$ determines a covariant derivative

$$\widetilde{\nabla} : \Gamma(\mathbb{S}) \to \Gamma(T^*M \otimes \mathbb{S})$$

given by

$$\widetilde{\nabla}\psi = d\psi + \kappa^T_{*1}(\widetilde{\omega \times (A,B)})\psi$$

where $\kappa^T_{*1} : \mathfrak{spin}^T(n) \to End(\Delta_n)$ is the derivative of κ at the identity $1 \in Spin^T(n)$. It can be also shown that

$$\kappa^T_{*1}(e_\alpha e_\beta, \lambda i, \mu i) = \kappa(e_\alpha e_\beta) + (2\lambda i + \mu i)Id$$

where λ and μ are any real numbers and κ is the spin representation of the group Spin(n).

Now we give the local formulas for connections. Fix a section $s : U \to P_{S^1 \times S^1}$ of the principal bundle $P_{S^1 \times S^1}$. Then, we obtain the local connection form

$$(A^s, B^s) : TU \to i\mathbb{R} \oplus i\mathbb{R}$$

where $A^s, B^s : TU \to i\mathbb{R}$. $e \times s : U \to P_{SO(n)} \times P_{S^1 \times S^1}$ is a local section of the fiber product bundle $P_{SO(n)} \times P_{S^1 \times S^1}$. $\widetilde{e \times s}$ is a lift of this section to the

two-fold covering $\Pi : P_{Spin^T(n)} \to P_{SO(n)} \times P_{S^1 \times S^1}$. The local connection form $\widetilde{\omega \times (A,B)}^{\widetilde{(e\times s)}}$ on the principal bundle $P_{Spin^T(n)}$ is given by the formula

$$\widetilde{\omega \times (A,B)}^{\widetilde{(e\times s)}} = \left(\frac{1}{2} \sum_{i<j} g(\nabla e_i, e_j) e_i e_j, \frac{1}{2} A^s, B^s - \frac{1}{2} A^s \right)$$

Hence, this connection form induces a connection $\widetilde{\nabla}$ on the spinor bundle $\mathbb{S}$. We can locally describe $\widetilde{\nabla}$ by

$$\widetilde{\nabla}_X \psi = d\psi(X) + \frac{1}{2} \sum_{i<j} g(\nabla_X e_i, e_j) e_i e_j \psi + \frac{1}{2} A^s \psi + B^s \psi \qquad (1)$$

where $\psi : U \to \Delta_n$ is a section of the spinor bundle $\mathbb{S}$.

Definition 7 *The first order differential operator*

$$D = \mu \circ \widetilde{\nabla} : \Gamma(\mathbb{S}) \xrightarrow{\widetilde{\nabla}} \Gamma(T^*M \otimes \mathbb{S}) \xrightarrow{\mu} \Gamma(\mathbb{S})$$

where μ denotes Clifford multiplication, is called the Dirac operator.

The Dirac operator D is locally given by

$$D\psi = \sum_{i=1}^{n} e_i \cdot \widetilde{\nabla}_{e_i} \psi \qquad (2)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on the manifold M.

The Dirac operator has the following property:

Theorem 8 *Let f be a smooth function and $\psi \in \Gamma(\mathbb{S})$ be a spinor field. Then,*

$$D(f \cdot \psi) = (gradf \cdot \psi) + fD\psi.$$

Proof By using the definition of the Dirac operator D we can compute $D(f \cdot \psi)$ as follows:

$$\begin{aligned} D(f \cdot \psi) &= \sum_{i=1}^{n} e_i \cdot \widetilde{\nabla}_{e_i} (f \cdot \psi) \\ &= \sum_{i=1}^{n} e_i \cdot (e_i(f) \cdot \psi + f \widetilde{\nabla}_{e_i} \psi) \\ &= \sum_{i=1}^{n} e_i(f) e_i \cdot \psi + f \sum_{i=1}^{n} e_i \cdot \widetilde{\nabla}_{e_i} \psi \\ &= (gradf) \cdot \psi + fD\psi \end{aligned}$$

□

Now we can define the Laplace operator on the spinor bundle $\mathbb{S}$.

Definition 9 *Let $\psi \in \Gamma(\mathbb{S})$ be a spinor field. The Laplace operator Δ on spinors is defined by*

$$\Delta \psi = -\sum_{i=1}^{n} \left(\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} \psi + div(e_i) \widetilde{\nabla}_{e_i} \psi \right). \qquad (3)$$

4.1 Schrödinger-Lichnerowicz type formula

The square D^2 of the Dirac operator and the Laplace operator Δ are second order differential operators. We derive Schrödinger-Lichnerowicz type formula by computing their difference $D^2 - \Delta$.

The curvature $R^{\mathbb{S}}$ of the spinor covariant derivative $\widetilde{\nabla}$ is an $End(\mathbb{S})$ valued 2−form by

$$R^{\mathbb{S}}(X,Y)\psi = \widetilde{\nabla}_X\widetilde{\nabla}_Y\psi - \widetilde{\nabla}_Y\widetilde{\nabla}_X\psi - \widetilde{\nabla}_{[X,Y]}\psi$$

where $\psi \in \Gamma(\mathbb{S})$ and $X, Y \in \Gamma(TM)$. Now we want to describe $R^{\mathbb{S}}$ in terms of the curvature tensor R.

Let $\Omega^\omega : TP_{SO(n)} \times TP_{SO(n)} \to \mathfrak{so}(n)$ be the curvature form of the Levi-Civita connection with the components

$$\Omega^\omega = \sum_{i<j} \Omega_{ij} E_{ij}$$

where $\Omega_{ij} : TP_{SO(n)} \times TP_{SO(n)} \to \mathbb{R}$. The commutative diagram defining the connection $\omega \times \widetilde{(A,B)}$ implies that the curvature form of $\omega \times \widetilde{(A,B)}$ is

$$\Omega^{\omega\times\widetilde{(A,B)}} = \frac{1}{2}\sum_{i<j}\Pi^*(\Omega_{ij})e_ie_j \oplus \frac{1}{2}\Pi^*(dA) \oplus \Pi^*(dB).$$

Hence the 2−form $R^{\mathbb{S}}$ with values in the spinor bundle $\mathbb{S}$ is obtained by the following formula:

$$R^{\mathbb{S}}(.,.)\psi = \frac{1}{2}\sum_{i<j}\Omega_{ij}e_ie_j\cdot\psi + \frac{1}{2}dA\cdot\psi + dB\cdot\psi.$$

Let $\{e_1,\ldots,e_n\}$ be orthonormal frame field, $\Omega_{ij}(X,Y) = g(R(X,Y)e_i,e_j)$ the components of the curvature form of the Levi-Civita connection, $X = \sum_{k=1}^{n} X^k e_k$ and $Y = \sum_{l=1}^{n} Y^l e_l$ be vector fields on the Riemannian manifold M. Then we have

$$\begin{aligned}
\Omega_{ij}(X,Y) &= g(R(X,Y)e_i,e_j) \\
&= \sum_{k,l=1}^{n} R_{klij}X^kY^l \\
&= \sum_{k,l=1}^{n} R_{klij}e^k(X)e^l(Y) \\
&= \frac{1}{2}\sum_{k,l=1}^{n} R_{klij}(e^k\wedge e^l)(X,Y).
\end{aligned}$$

where $\{e^1,\ldots,e^n\}$ is the frame dual to $\{e_1,\ldots,e_n\}$. Thus, we obtain the following local formula for the curvature form

$$\Omega^{\omega\times\widetilde{(A,B)}} = \frac{1}{4}\sum_{i<j}\sum_{k,l=1}^{n} R_{klij}(e^k\wedge e^l)e_ie_j + \frac{1}{2}dA + dB$$

and the 2-form $R^{\mathbb{S}}(.,.)$ is calculated as follows:

$$R^{\mathbb{S}}(.,.)\psi = \frac{1}{4}\sum_{i<j}\sum_{k,l=1}^{n} R_{klij}(e^k \wedge e^l)e_i e_j \cdot \psi + \frac{1}{2}dA\cdot\psi + dB\cdot\psi.$$

By using the above properties of the curvature form $R^{\mathbb{S}}$ on spinor bundle $\mathbb{S}$ we deduce the following result:

Proposition 10 *Let Ric be the Ricci tensor. Then, the following relation holds:*

$$\sum_{\alpha=1}^{n} e_\alpha \cdot R^{\mathbb{S}}(X, e_\alpha)\psi = -\frac{1}{2}Ric(X)\cdot\psi + \frac{1}{2}(X \lrcorner dA)\cdot\psi + (X \lrcorner dB)\cdot\psi \tag{4}$$

Proof In [2] it is proved the following relation:

$$\sum_{\alpha=1}^{n}\sum_{i<j}\sum_{k,l=1}^{n} R_{klij}(e^k \wedge e^l)e_\alpha e_i e_j \cdot \psi = -2Ric(X)\cdot\psi \tag{5}$$

It can be easily seen the following two relations:

$$\sum_{\alpha=1}^{n} e_\alpha \cdot dA(X, e_\alpha)\cdot\psi = (X \lrcorner dA)\cdot\psi \tag{6}$$

and

$$\sum_{\alpha=1}^{n} e_\alpha \cdot dB(X, e_\alpha)\cdot\psi = (X \lrcorner dB)\cdot\psi. \tag{7}$$

Then, using (5), (6) and (7), we obtain the claimed equivalence. □

Now, we derive Schrödinger-Lichnerowicz-type formula in the following way:

Proposition 11 *Let s be scalar curvature of the Riemannian manifold and let $dA = \Omega^A$ and $dB = \Omega^B$ be the imaginary-valued $2-$forms of the connections (A, B) in the $(S^1 \times S^1)-$bundle associated with Spin^T structure. Then, we have the following formula:*

$$D^2\psi = \Delta\psi + \frac{s}{4}\psi + \frac{1}{2}dA\cdot\psi + dB\cdot\psi.$$

Proof

$$\begin{aligned} D^2\psi &= \sum_{i,j} e_i \cdot \widetilde{\nabla}_{e_i}(e_j \cdot \widetilde{\nabla}_{e_j}\psi) \\ &= \sum_{i,j} e_i \cdot \nabla_{e_i} e_j \cdot \widetilde{\nabla}_{e_j}\psi + e_i e_j \cdot \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}\psi \\ &= \sum_{i,j,k} g(\nabla_{e_i} e_j, e_k) e_i e_k \cdot \widetilde{\nabla}_{e_j}\psi + \sum_{i,j} e_i e_j \cdot \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}\psi \\ &= \Delta\psi + \sum_{j,i\neq k} g(\nabla_{e_i} e_j, e_k) e_i e_k \cdot \widetilde{\nabla}_{e_j}\psi + \sum_{i\neq j} e_i e_j \cdot \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}\psi \end{aligned} \tag{8}$$

Now we can calculate the following sum:

$$\begin{aligned}\sum_{i\neq k} g(\nabla_{e_i}e_j, e_k)e_ie_k &= -\sum_{i\neq k} g(e_j, \nabla_{e_i}e_k)e_ie_k \\ &= -\sum_{i<k} g(e_j, \nabla_{e_i}e_k - \nabla_{e_k}e_i)e_ie_k \\ &= \sum_{i<k} g(e_j, [e_k, e_i])e_ie_k\end{aligned}$$

From (8) we get

$$\begin{aligned}D^2\psi &= \Delta\psi + \sum_{j,i<k} g(e_j, [e_k, e_i])e_ie_k\widetilde{\nabla}_{e_j}\psi + \sum_{i<j} e_ie_j\cdot(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}\psi - \widetilde{\nabla}_{e_j}\widetilde{\nabla}_{e_i}\psi) \\ &= \Delta\psi + \sum_{i<j} e_ie_j(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}\psi - \widetilde{\nabla}_{e_j}\widetilde{\nabla}_{e_i}\psi - \widetilde{\nabla}_{[e_i,e_j]}\psi) \\ &= \Delta\psi + \frac{1}{2}\sum_{i,j} e_ie_jR^{\mathbb{S}}(e_i, e_j)\psi.\end{aligned}$$

Using the identity (4) and multiplying by e_i we deduce

$$\begin{aligned}D^2\psi &= \Delta\psi - \frac{1}{4}\sum_i e_iRic(e_i)\cdot\psi + \frac{1}{4}\sum_i e_i\cdot(e_i \lrcorner\, dA)\cdot\psi + \frac{1}{2}\sum_i e_i\cdot(e_i \lrcorner\, dB)\cdot\psi \\ &= \Delta\psi + \frac{s}{4}\psi + \frac{1}{2}dA\cdot\psi + dB\cdot\psi.\end{aligned}$$

□

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